

# Exact sampling of first passage event of certain symmetric Lévy processes with unbounded variation

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June 22, 2016

## Abstract

We show that exact sampling of the first passage event can be done for a Lévy process with unbounded variation, if the process can be embedded in a subordinated standard Brownian motion. By sampling a series of first exit events of the Brownian motion and first passage events of the subordinator, the first passage event of interest can be obtained. The sampling of the first exit time and pre-exit location of the Brownian motion may be of independent interest.

*Keywords and phrases.* First exit; Lévy process; unbounded variation; subordinator

2000 Mathematics Subject Classifications: Primary 60G51; Secondary 60E07.

## 1 Introduction

The first exit event of a Lévy process is an intensively studied subject in probability [1, 2, 4, 10, 11, 13, 14, 17]. Despite numerous deep theoretical results on the subject, exact sampling of the first passage event remains challenging especially for processes with infinite Lévy measures. Recently, in [6, 7], it was shown that exact sampling can be done for a wide range of Lévy processes with bounded variation. The methods of [6, 7] rely on the decomposition of a process as the difference of two independent subordinators. However, no such decomposition exists for Lévy processes with unbounded variation. In this paper, we show that exact sampling of the first exit event can be achieved for several important classes of such processes, a primary example being those that have a symmetric truncated  $\alpha$ -stable Lévy measure with  $\alpha \in [1, 2)$  and with or without a Brownian component. One of the main ingredients of the method is the sampling of the time when a Brownian motion first exits an interval and a pre-exit location of the Brownian motion, which may be of interest in its own right.

The Lévy processes covered by the paper are among those that can be embedded into a subordinated Brownian motion ([17], Chapter 30). Similar to [7], the general idea is to sample certain first exit events of the subordinated Brownian motion and extract from them the part that belong to the Lévy process under consideration. However, care is required to deal with unbounded variation. Due to the structure of the subordinated Brownian motion, the sampling of the first exit event can be addressed by attacking two issues in tandem. The first one is the sampling of the first passage event of the subordinator underlying the subordinated Brownian motion. For this, the results in [7] can be directly used. The second issue is the sampling of the first exit time of a Brownian motion and its value at a pre-exit time point given the time and location of the first exit. For this, a huge number of known results can be used (cf. [5, 16]).

Section 3 presents the main sampling procedure. First, the scheme of the procedure is illustrated in Section 3.1. Then, in Section 3.2, the procedure is formalized as Algorithm 1. In Section 3.3, some examples are given. Since the sampling of the first passage time of the subordinator involved was addressed previously [7], after Section 3, the discussion is mostly dedicated to the sampling for

the Brownian motion. In Section 4, we consider the sampling of the first exit time of the Brownian motion by exploiting its well known distributional properties [5]. We then consider the sampling of the value of the Brownian motion at a time point before its first exit. In Section 5, we obtain some useful results on the distribution of the pre-exit value of the Brownian motion, which we have not been able to find in the literature. In Section 6, the sampling of the pre-exit value is considered. The subtlety here is the handling of the many negative terms in the series expansion of the density function of the pre-exit value. Finally, some comments are made in section 7.

## 2 Preliminaries

For  $c \in (0, 1)$ , denote by  $\text{Geo}(c)$  the geometric distribution on  $\{0, 1, 2, \dots\}$  with probability mass function (p.m.f.)  $(1 - c)c^k$ . For  $\theta > 0$ , denote by  $\text{Gamma}(\theta)$  the Gamma distribution on  $(0, \infty)$  with probability density function (p.d.f.)  $\mathbf{1}\{x > 0\} x^{\theta-1} e^{-x} / \Gamma(\theta)$ . Denote by  $\phi_c(x)$  the p.d.f. of  $N(0, c)$ , the normal distribution with mean 0 and variance  $c$ .

### 2.1 Lévy processes

Let  $X = (X_t)_{t \geq 0}$  be a Lévy process. Then  $\mathbb{E}[e^{i\lambda X_t}] = e^{-t\Psi_X(\lambda)}$ ,  $t \geq 0$ ,  $\lambda \in \mathbb{R}$ , where for some  $c, \sigma \in \mathbb{R}$  and measure  $\nu$  on  $\mathbb{R} \setminus \{0\}$  which satisfies  $\int \min(x^2, 1) \nu(dx) < \infty$ ,

$$\Psi_X(\lambda) = -ic\lambda + \sigma^2 \lambda^2 / 2 + \int (1 - e^{i\lambda x} + i\lambda x \mathbf{1}\{|x| < 1\}) \nu(dx).$$

$\Psi_X$ ,  $c$ ,  $\sigma$ , and  $\nu$  are called the characteristic exponent, linear coefficient, Brownian coefficient, and Lévy measure of  $X$ , respectively. The Radon-Nikodym derivative  $\nu(dx)/dx$ , provided it exists, is called the Lévy density of  $X$ . If  $\sigma \neq 0$  or  $\int \min(|x|, 1) \nu(dx) = \infty$ , then  $X$  is said to have infinite variation, otherwise it is said to have finite variation. In the latter case  $\Psi_X$  can be written as

$$\Psi_X(\lambda) = -i\delta\lambda + \int (1 - e^{i\lambda x}) \nu(dx),$$

with  $\delta$  called the drift coefficient of  $X$ . A necessary and sufficient condition for  $X$  to be nondecreasing is that it has finite variation with  $\delta \geq 0$  and  $\nu((-\infty, 0)) = 0$ . In this case,  $X$  is called a subordinator. Furthermore, if  $Y = (Y_t)_{t \geq 0}$  is a process independent of  $X$ , then  $Y_X = (Y_{X_t})_{t \geq 0}$  is called a subordinated process. In particular, if  $Y$  is a standard Brownian motion, i.e., a Lévy process with  $\Psi_Y(\lambda) = \lambda^2/2$ , then  $Y_X$  is a Lévy process with linear coefficient 0, Brownian coefficient  $\sqrt{\delta}$ , and Lévy density  $\int_0^\infty \phi_s(\cdot) \nu(ds)$  ([17], Theorem 30.1).

For any open or closed set  $A \subset \mathbb{R}$ , according to Blumenthal 0-1 law, either 0 is regular for  $A$  (with respect to  $X$ ), i.e.,  $\varsigma := \inf\{t > 0 : X_t \in A\} = 0$  a.s., or 0 is irregular for  $A$ , i.e.,  $\varsigma > 0$  a.s. ([17], p. 313). It is easy to see that if  $X$  is not a compound Poisson process and is symmetric, i.e.,  $X \sim -X$ , then 0 is regular for both half-lines  $(-\infty, 0)$  and  $(0, \infty)$ . As a passing remark, if  $X$  has infinite variation, then 0 is regular for both half-lines ([2], p. 167), while if  $X$  has finite variation, Bertoin's test provides a necessary and sufficient condition for 0 to be regular for a half-line ([3]; [9], Theorem 6.22).

### 2.2 First hitting times of a Brownian motion

Let  $B = (B_t)_{t \geq 0}$  such that  $B_t - B_0$  is a standard Brownian motion. For  $a \in \mathbb{R}$ , let

$$\tau_a = \inf\{t : B_t = a\},$$

while for  $a > 0$ , let

$$\eta_a = \inf\{t : B_t \in \{-a, a\}\} = \min(\tau_{-a}, \tau_a).$$

Given  $x \in \mathbb{R}$ , denote by  $P^x$  the probability measure under which  $B_0 \equiv x$  and by  $E^x$  the associated expectation. It is well known that under  $P^x$  the p.d.f. of  $\tau_a$  is  $f_{a-x}$ , where

$$f_a(t) = |a|t^{-1}\phi_t(a) = -\phi'_t(|a|) = \frac{|a|e^{-a^2/(2t)}}{\sqrt{2\pi t^3/2}}, \quad (2.1)$$

while for  $a > 0$  and  $x \in (-a, a)$ , the p.d.f. of  $\eta_a$ , denoted by  $p_a(t, x)$ , has series expressions

$$p_a(t, x) = \sum_{k=0}^{\infty} (-1)^k [f_{2ka+a-x}(t) + f_{2ka+a+x}(t)]. \quad (2.2)$$

$$= \frac{\pi}{2a^2} \sum_{k=0}^{\infty} (-1)^k (2k+1) \exp\left\{-\frac{(2k+1)^2 \pi^2 t}{8a^2}\right\} \cos \frac{(2k+1)\pi x}{2a}. \quad (2.3)$$

The series in (2.2) converges rapidly for small  $t > 0$  but slowly for large  $t > 0$ , while the one in (2.3) has the opposite property. Eq. (2.3) is due to the following fact ([16], §7.4). Let  $U$  be a bounded and connected open set and  $g \in C(U)$ . If  $u(t, x) \in C^2((0, \infty) \times U)$  is a bounded solution to the heat equation  $\partial_t u = (1/2)\partial_{xx} u$  with initial condition  $\lim_{(t,x) \rightarrow (0,x_0)} u(t, x) = g(x_0)$  for all  $x_0 \in U$  and Dirichlet boundary condition  $\lim_{(t,x) \rightarrow (t_0, x_0)} u(t, x) = 0$  for all  $t_0 > 0$  and  $x_0 \in \partial U$ , then

$$u(t, x) = E^x[g(B_t)\mathbf{1}\{t < \eta_U\}], \quad \text{where } \eta_U = \inf\{t : B_t \notin U\}.$$

Calculating  $E^x[g(B_t)\mathbf{1}\{t < \eta_U\}]$  then boils down to solving the heat equation with the specified initial and boundary conditions. In particular, if  $U = (-a, a)$ , by separating the variables  $t$  and  $x$  and considering the eigenfunctions of  $(1/2)\partial_{xx}$  and those of  $\partial_t$  with the same eigenvalues,

$$u(t, x) = \sum_{k=0}^{\infty} \alpha_k \exp\left\{-\frac{k^2 \pi^2 t}{8a^2}\right\} \varphi_k\left(\frac{x}{2a}\right), \quad (2.4)$$

where  $\varphi_k(x)$  is  $\cos(k\pi x)$  for odd  $k$  and  $\sin(k\pi x)$  for even  $k$ , and  $\alpha_k = a^{-1} \int_{-a}^a g(x)\varphi_k(x) dx$ . To get Eq. (2.3), apply (2.4) to  $g(x) \equiv 1$  to yield  $P^x\{\eta_a > t\}$ , and then differentiate the result in  $t$ .

### 3 Main results

Let  $X$  be a symmetric Lévy process. Suppose the Brownian coefficient of  $X$  is  $\delta \geq 0$  and its Lévy density is

$$\lambda(x) = \lambda_0(x)\mathbf{1}\{|x| < r\}, \quad 0 < r \leq \infty,$$

such that  $\lambda_0(x)$  is the Lévy density of a subordinated process  $Z = B_S$ , where  $B$  is a standard Brownian motion and  $S$  a subordinator with drift coefficient  $\delta^2$  independent of  $B$ . Then  $Z$  is symmetric and its Brownian coefficient is  $\delta$  as well.

Given interval  $I = (b, c)$  with  $-\infty \leq b < 0 < c \leq \infty$ , the first exit time of  $X$  out of  $I$  is

$$T_I = \inf\{t > 0 : X_t \notin I\}.$$

The value and jump of  $X$  at the time of exit are also important information. We shall consider the sampling of the triplet  $(T_I, X_{T_I-}, X_{T_I})$ , where for  $t > 0$ ,  $X_{t-}$  is the left limit of  $X$  at  $t$ .

By right-continuity of  $X_t$ ,  $T_I > 0$ . As long as  $X_t \not\equiv 0$  and  $\min(|b|, c) < \infty$ ,  $T_I$  is finite. This is because either i)  $\lim X_t = \infty$  a.s., or ii)  $\lim X_t = -\infty$  a.s., or iii)  $\limsup X_t = -\liminf X_t = \infty$  a.s. ([2], Theorem VI.16). Since  $X$  is symmetric, iii) must hold, so  $T_I < \infty$ . As a passing remark, for any Lévy process, Erickson's test provides a necessary and sufficient condition on which of the three cases holds ([9], Theorem 4.15 and p. 64).

In this section, it is always assumed that

$$X \text{ is not a compound Poisson process,} \quad (3.1)$$

which is equivalent to  $S$  not being a compound Poisson subordinator. Since  $X$  is symmetric, then with respect to  $X$ ,  $c$  is regular for  $(c, \infty)$  and  $b$  is regular for  $(-\infty, b)$ , so

$$T_I = \inf\{t > 0 : X_t \notin [b, c]\}. \quad (3.2)$$

In particular, if  $0 < c < \infty$ ,  $T_{(-\infty, c)}$  is the first passage time of  $X$  across  $c$ , i.e.,

$$T_{(-\infty, c)} = \inf\{t > 0 : X_t > c\}.$$

By definition,  $Z_{t-} = \lim_{u \rightarrow t-} B_{S_u}$ . Under assumption (3.1),

$$Z_{t-} = B_{S_{t-}}. \quad (3.3)$$

Indeed, as  $u \rightarrow t-$ , since  $S$  is not a compound Poisson process and hence strictly increasing,  $S_u \rightarrow s := S_{t-}$ , given  $B_{S_u} \rightarrow B_{s-} = B_s$  by continuity of  $B$ . Then (3.3) follows.

### 3.1 Description

To sample  $(T_I, X_{T_I-}, X_{T_I})$ , the approach is to embed  $X$  into  $Z = B_S$  and exploit a sequence of hitting or passage events of  $B$  and  $S$ . By embedding it means that, by identifying  $Z$  with  $X + V$ ,  $X$  is a part of  $Z$ , where  $V$  is a compound Poisson process with Lévy density  $\lambda_0(x)\mathbf{1}\{|x| \geq r\}$  independent of  $X$ . Equivalently,

$$X_t = Z_t - \sum_{s \leq t} \Delta Z_s \mathbf{1}\{|\Delta Z_s| \geq r\}, \quad (3.4)$$

where  $\Delta Z_s = Z_s - Z_{s-}$  is the jump of  $Z$  at  $s$ . The issue is how to identify the jumps of  $V$  so that random variables purely due to  $X$  can be extracted from those of  $Z$ .

Figure 1 illustrates the idea. Suppose  $(-r/2, r/2) \subset I$ . At time  $\eta_{r/2}$ ,  $B$  hits the boundary of  $(-r/2, r/2)$ , say at  $r/2$ . Then

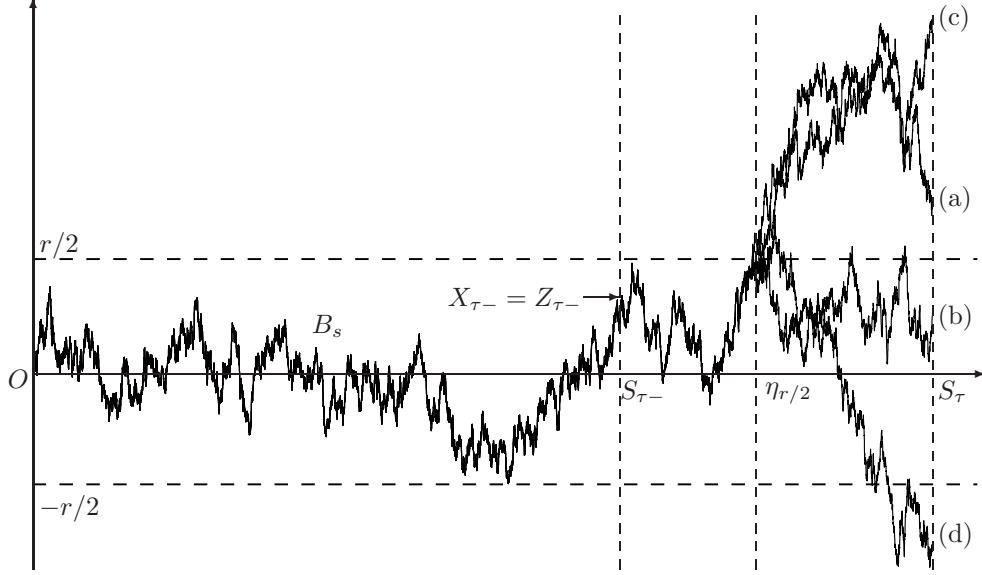
$$|B_{s_2} - B_{s_1}| < r \quad \text{for all } 0 < s_1 < s_2 \leq \eta_{r/2}. \quad (3.5)$$

Let

$$\tau = \inf\{t > 0 : S_t > \eta_{r/2}\}.$$

From (3.3) and (3.5), it follows that for  $0 < t < \tau$ ,  $|\Delta Z_t| < r$ . Thus, in  $[0, \tau)$ ,  $Z$  only has jumps of size strictly less than  $r$ . Then  $X_t = Z_t$  on  $[0, \tau)$  and the first jump in  $V$  can only appear at  $\tau$ . Figure 1 shows a scenario where  $S$  has a jump at  $\eta_{r/2}$ . Because the potential measure of  $S$  is diffuse,  $S_{\tau-} < \eta_{r/2} < S_\tau$  ([2], Propositions I.15 and III.2). Then  $X_{\tau-} = Z_{\tau-} = B_{S_{\tau-}}$  and by (3.3),

Figure 1:  $Z = B_S$ . (a)  $r/2 \leq Z_\tau \leq Z_{\tau-} + r$ , so  $X_\tau = Z_\tau$  and  $\tau$  is the first exit time of  $X$  and  $Z$  out of  $(-r/2, r/2)$ ; (b)  $-r/2 \leq Z_\tau \leq r/2$ , so  $X_\tau = Z_\tau$ , but  $\tau$  is not an exit time of  $X$  out of  $(-r/2, r/2)$ ; (c)  $Z_\tau > Z_{\tau-} + r$ , so  $X_\tau = X_{\tau-}$ ,  $\tau$  is not the first exit time of  $X$  out of  $(-r/2, r/2)$  but is that of  $Z$  (from the top); (d)  $Z_\tau < -r/2$



$\Delta Z_\tau = B_{S_\tau} - B_{S_{\tau-}}$ . If  $|\Delta Z_\tau| < r$ , then the jump belongs to  $X$ , giving  $X_\tau = Z_\tau$ . This is the case in (a) and (b) in Figure 1. In (a), since  $X_\tau \geq r/2$ ,  $\tau$  is the first time of  $X$  out of  $(-r/2, r/2)$ , whereas in (b), it is not. If  $|\Delta Z_\tau| \geq r$ , then the jump belongs to  $V$ , giving  $X_\tau = X_{\tau-}$ . This is the case in (c) in Figure 1, where  $\tau$  is the first exit time of  $Z$  out of  $(-r/2, r/2)$  but not that of  $X$ . Also, as (d) shows, even if  $B$  first exits  $(-r/2, r/2)$  at the top boundary point,  $Z$  and  $X$  may first exit at the lower boundary.

From the above description, it is seen that the following quantities have to be sampled in sequel,

- a)  $\eta_{r/2}$ , the first hitting time of  $B$  at  $\pm r/2$ ;
- b) the triplet  $(\tau, S_{\tau-}, S_\tau)$ , where  $\tau$  the first passage time of  $S$  across  $\eta_{r/2}$ ; and
- c)  $B_{S_{\tau-}}$  and  $B_{S_\tau}$ .

Because  $S$  and  $B$  are independent, b) boils down to the sampling of the first passage of  $S$  across any fixed level. This is addressed in [7] for several important classes of subordinators. Items a) and c) will be considered in following sections. The sampling of  $B_{S_{\tau-}}$  boils down to that of  $B_T$  conditional on  $(\eta_{r/2}, B_{\eta_{r/2}}) = (T + t, \pm r/2)$  for fixed  $T, t > 0$ . On the other hand, by the strong Markov property of Brownian motion,  $B_{S_\tau}$  can be simply sampled from a normal distribution.

Figure 1 just illustrates a single iteration of the sampling procedure. If at the end of the iteration,  $X$  has yet exited  $I$ , then the procedure is renewed at time  $\tau$ . The iteration continues until an exit occurs. Note that  $\tau$  is not a stopping time of  $X$  or  $Z$  as it depends on  $\eta_{r/2}$ , information not available via  $X$  or  $Z$ . However, conditional on  $B$ ,  $\tau$  is a stopping time of  $S$ , justifying iterating by renewal. Indeed, the procedure can be thought of as one with the entire path of  $B$  being sampled in advance and  $S$  being the only random process during the run. In this setting, the sampling of  $\eta_{r/2}$ ,  $Z_{\tau-}$  and  $Z_\tau$  can be regarded as a subroutine to retrieve data from the path of  $B$ .

There are two simpler cases not covered so far. First, if  $S$  has a positive drift, then it may creep across  $\eta_{r/2}$ , i.e.,  $S_{\tau-} = S_\tau = \eta_{r/2}$  ([2], Theorem III.5). Clearly in this case  $X_{\tau-} = X_\tau$  is equal to the

point where  $B$  exits  $(-r/2, r/2)$ . Second, suppose we wish to sample  $(\varsigma, X_{\varsigma-}, X_{\varsigma})$  instead, where  $\varsigma = \min(T_I, T_0)$  with  $T_0 < \infty$  a fixed terminal point. This allows, for example, the sampling of  $X_{T_0}$  when  $I = \mathbb{R}$ . If  $T_0 < \tau$ , then  $S_{T_0-} = S_{T_0}$  instead of  $S_{\tau-}$  and  $S_{\tau}$  should be sampled conditional on  $S_{T_0} < \eta_{r/2}$  (cf. [7]) and then  $X_{T_0} = Z_{T_0} = B_{S_{T_0}}$  is sampled conditional on  $(\eta_{r/2}, B_{\eta_{r/2}})$ .

### 3.2 Formal procedure

The description in Section 3.1 is formalized as Algorithm 1 with additional detail taken into account. In the procedure,  $I = (b, c)$  and  $\varsigma$  is defined at the end of Section 3.1. It is quite routine to extend the procedure to sample the first exit event of  $X + Y$ , where  $Y$  is a compound Poisson process independent of  $X$  (cf. [7]). For brevity, the detail of the extension is omitted.

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#### Algorithm 1 Sampling $(\varsigma, X_{\varsigma-}, X_{\varsigma})$

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**Require:**  $-b, c, r, T_0$  all in  $(0, \infty]$  such that if  $|b| = c = \infty$  then  $r < \infty$  and  $T_0 < \infty$

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1:  $T \leftarrow 0, W \leftarrow 0$ 
2: repeat
3:    $a \leftarrow \min(r/2, W - b, c - W)$ 
4:   Sample  $h$  from the distribution of  $\eta_a$  and sample  $y$  from  $\{-a, a\}$  uniformly
5:   Sample  $(t, s_-, s_+)$  from the distribution of  $(\theta, S_{\theta-}, S_{\theta})$ , where  $\theta = \min(\tau, T_0 - T)$  with  $\tau = \inf\{t > 0 : S_t > h\}$ 
6:   Sample  $x$  from the conditional distribution of  $B_{s_-}$  given  $(\eta_a, B_{\eta_a}) = (h, y)$ 
7:   if  $s_- < s_+$  then
8:     Sample  $u$  from  $N(0, s_+ - s_-)$ 
9:      $d \leftarrow y - x + u, D \leftarrow d\mathbf{1}\{|d| \leq r\}$ 
10:  else
11:     $D \leftarrow 0$ 
12:  end if
13:   $T \leftarrow T + t, W \leftarrow W + x + D$ 
14: until  $T = T_0$  or  $W \notin (b, c)$ 
15: return  $(T, W - D, W)$ 

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In Algorithm 1,  $T$  is a value such that  $X_t \in (b, c)$  for all  $t \in (0, T)$  and  $W = X_T$ . In each iteration  $(T, W)$  is updated following the description in Section 3.1. Some explanations are in order. The value of  $a$  on line 3 makes sure the interval  $(W - a, W + a)$  is in  $(b, c)$ . Since  $B$  is a standard Brownian motion,  $y$  on line 4 follows the distribution of  $B_{\eta_a}$  conditional on  $\eta_a$ . If  $s_- < s_+$ , then as explained in Section 3.1,  $t < T_0 - T$  and  $s_- < h < s_+$ . By the strong Markov property of  $B$  and the independence between  $B$  and  $S$ , conditional on  $S_{\theta}$ ,  $B_{S_{\theta}} - B_{\eta_a}$  is independent of  $(B_t)_{t \leq \eta_a}$  and follows  $N(0, B_{S_{\theta}} - B_{\eta_a})$ . Thus  $u$  on line 8 follows the distribution of  $B_{S_{\theta}} - B_{\eta_a}$ , and  $d$  and  $D$  on line 9 follow the joint distribution of  $\Delta Z_{\tau}$  and  $\Delta X_{\tau}$ . On the other hand, if  $s_- = s_+$ , then either  $s_- = h$  or  $T_0 - T < t$ , resulting in no jump and line 11. Line 13 updates the values of  $T$  and  $W$ . It is clear that the iteration stops only when  $T = T_0$  or  $X_T \in (b, c)$ , i.e.  $T = \varsigma$ . When the iteration stops, since  $D = \Delta X_T$ ,  $(T, W - D, W)$  follows the distribution of  $(\varsigma, X_{\varsigma-}, X_{\varsigma})$ .

**Theorem 1.** *The iteration in Algorithm 1 eventually stops a.s. and its output follows the distribution of  $(\varsigma, X_{\varsigma-}, X_{\varsigma})$ .*

*Proof.* It suffices to show that the iteration eventually stops a.s. Let  $t_0 = 0, w_0 = 0$ , and for  $n \geq 1$ ,  $t_n$  and  $w_n$  the values of  $T$  and  $W$  at the end of the  $n^{\text{th}}$  iteration, respectively. Let  $s_n = S_{t_n}$  and

$z_n = Z_{t_n}$ . It is easy to see that both  $t_n$  and  $s_n$  are strictly increasing and  $t_n \leq \varsigma < \infty$ . Define event

$$E = \{\text{the iteration in Algorithm 1 does not stop}\}.$$

In  $E$ , there are infinitely many  $t_n$ . Let  $t_* = \lim t_n$  and  $s_* = \lim s_n$ . Since conditional on  $B$ ,  $t_n$  are stopping times of  $S$ ,  $s_* = S_{t_*}$  a.s. ([2], Proposition I.7). Then by continuity of  $B$ ,

$$\sup_{t_n \leq t < t' \leq t_*} |Z_{t'} - Z_t| \leq 2 \sup_{t_n \leq t \leq t_*} |Z_t - B_{s_*}| \leq 2 \sup_{s_n \leq s \leq s_*} |B_s - B_{s_*}| \rightarrow 0, \quad n \rightarrow \infty.$$

As a result,  $z_n \rightarrow B_{s_*}$  and there is  $N \geq 0$  such that  $|\Delta Z_t| < r$  on  $[t_N, t_*]$ . By renewal argument, we can assume  $N = 0$  without loss of generality. Then  $X_t = Z_t$  on  $[0, t_*]$ , in particular,

$$w_n = z_n \rightarrow B_{s_*}. \quad (3.6)$$

On the other hand, let  $a_n$ ,  $h_n$ , and  $\tau_n$  be the values of  $a$ ,  $h$ , and  $\tau$  in the  $n^{\text{th}}$  iteration, respectively. Then  $t_n = t_{n-1} + \min(\tau_n, T_0 - t_{n-1})$ . In  $E$ ,  $t_n < T_0$ . Then  $\tau_n = t_n - t_{n-1} \rightarrow 0$ . Since  $\tau_n \sim \inf\{t > 0 : S_t > h_n\}$ , then  $h_n \rightarrow 0$ . Since  $h_n \sim \inf\{t > 0 : B_t > a_n\}$ , then  $a_n \rightarrow 0$ . Consequently,  $\min(w_n - b, c - w_n) \rightarrow 0$ . Combined with (3.6), this yields  $w_n = z_n$  either converges to  $b$  or to  $c$ . Without loss of generality, suppose the limit is  $c$ . By using renewal argument again, we can assume that  $c - w_n < w_n - b$  for all  $n \geq 0$ . Note that by renewing  $(X_t, Z_t)$  at  $t = t_{n-1}$ ,  $B_s$  is renewed at  $s = s_{n-1}$ . It follows that  $s_{n-1} + h_n$  is the first  $s > s_{n-1}$  such that  $B_s$  hits the boundary of  $(w_n - a_n, c)$  and, with  $S$  being non-compound Poisson,  $t_n$  is the first  $t \geq t_{n-1}$  such that  $S_t \geq s_{n-1} + h_n = S_{t_{n-1}} + h_n$ . Now  $B_{s_{n-1}+h_n}$  is either  $w_n - a_n$  or  $c$ , each with probability  $1/2$ . If  $B_{s_{n-1}+h_n} = c$ , then  $B_{s_n} - B_{s_{n-1}+h_n} = w_n - c < 0$ . By strong Markov property of  $B$ ,  $B_{s_n} - B_{s_{n-1}+h_n}$  are independent normal random variables with mean 0, possibly degenerate, so the probability that the procedure does *not* stop at the  $n^{\text{th}}$  iteration is at most  $\mathbb{P}\{B_{s_{n-1}+h_n} = w_n - a_n\} + \mathbb{P}\{B_{s_{n-1}+h_n} = c, z_n - c < 0\} = 3/4$ . It follows that the probability to have infinite iterations is 0. Then  $\mathbb{P}(E) = 0$ .  $\square$

### 3.3 Examples

**Example 1.** Let  $X$  be a symmetric Lévy process with Lévy density  $\lambda(x) = c\mathbf{1}\{0 < |x| < r\}x^{-\alpha-1}$ ,  $r \in (0, \infty)$ . If  $\alpha \in (0, 1)$  and the Brownian coefficient  $\delta$  of  $X$  is 0, then  $X$  has finite variation. In this case, it has been shown that  $(\varsigma, X_{\varsigma-}, X_{\varsigma})$  can be sampled exactly [6, 7]. On the other hand, if  $\delta \neq 0$  or  $\alpha \in [1, 2)$ ,  $X$  has infinite variation. In this case,  $X$  can be embedded into  $Z = B_S$  as in (3.4), where  $S$  is a subordinator with drift coefficient  $\delta^2$  and Lévy density  $c\mathbf{1}\{x > 0\}x^{-\alpha/2-1}/d_{\alpha/2}$ , where for  $z > 0$ ,  $d_z = \Gamma(z + 1/2)2^z/\sqrt{\pi}$ . The first passage event of  $S$  can be sampled exactly [6, 7]. By combining this with the results in following sections, Algorithm 1 can sample  $(\varsigma, X_{\varsigma-}, X_{\varsigma})$ .

**Example 2.** Suppose instead that  $S$  is a subordinator with exponentially tilted Lévy density  $c\mathbf{1}\{x > 0\}e^{-sx}x^{-\alpha/2-1}/d_{\alpha/2}$ , where  $s > 0$  and  $\alpha \geq 1$ . The first passage event of  $S$  can be sampled exactly [6, 7]. On the other hand, the Lévy density of  $B_S$  is

$$\begin{aligned} \lambda_0(x) &= \frac{c}{d_{\alpha/2}} \int_0^\infty \frac{e^{-x^2/2u}}{\sqrt{2\pi u}} e^{-su} u^{-\alpha/2-1} du \\ &= c\mathbf{1}\{x > 0\}x^{-\alpha-1} + \frac{c}{d_{\alpha/2}} \int_0^1 \frac{e^{-x^2/2u}}{\sqrt{2\pi u}} (e^{-su} - 1) u^{-\alpha/2-1} du + O(1) \\ &= c\mathbf{1}\{x > 0\}x^{-\alpha-1} - \frac{cs}{d_{\alpha/2}} \int_0^1 \frac{e^{-x^2/2u}}{\sqrt{2\pi}} u^{-\alpha/2-1/2} du + O(1 + s^2), \end{aligned}$$



where the implicit constant in  $O(1+s^2)$  only depends on  $(\alpha, c)$ . By variable substitute  $v = x^2/(2u)$ ,

$$\begin{aligned} \int_0^1 \frac{e^{-x^2/2u}}{\sqrt{2\pi}} u^{-\alpha/2-1/2} du &= \frac{1}{\sqrt{2\pi}} \left( \frac{2}{x^2} \right)^{\alpha/2-1/2} \int_{x^2/2}^{\infty} e^{-v} v^{\alpha/2-3/2} dv \\ &\sim \begin{cases} d_{\alpha/2-1} x^{-\alpha+1} & \alpha \in (1, 2) \\ -\ln|x|/\sqrt{2\pi} & \alpha = 1 \end{cases} \quad \text{as } x \rightarrow 0. \end{aligned}$$

Therefore, if  $\alpha \in (0, 1)$  and  $X$  has Lévy density  $\lambda(x) = c\mathbf{1}\{0 < |x| < r\}(1 - Cx^2)x^{-\alpha-1}$ , where  $C > 0$ , then by choosing  $s > 0$  large enough and  $r' \in (0, r]$  small enough,  $\lambda$  can be written as  $\lambda(x) = \mathbf{1}\{0 < |x| < r'\} \lambda_0(x) + \chi(x)$ , where  $\chi$  is the Lévy density of a compound Poisson process. Then, as noted just before Algorithm 1, a routine extension of the procedure is able to sample  $(\varsigma, X_{\varsigma-}, X_{\varsigma})$ . The case  $\alpha = 1$  can be similarly dealt with.

**Example 3.** Let  $X$  be a symmetric Lévy process with  $\lambda(x) = c\mathbf{1}\{0 < |x| < r\}e^{-x/\beta}/x$  and Brownian coefficient  $\delta$ . It has been shown that if  $\delta = 0$ , then  $(\varsigma, X_{\varsigma-}, X_{\varsigma})$  can be sampled exactly [6, 7]. Note that  $\lambda$  is a truncated version of the Lévy density of  $U - D$ , where  $U$  and  $D$  are independent Gamma processes with Lévy density  $\lambda_0(x) = c\mathbf{1}\{x > 0\}e^{-x/\beta}/x$ . It is well known that  $U - D \sim B_V$ , where  $V$  is a Gamma process with Lévy density  $c\mathbf{1}\{x > 0\}e^{-x/2\beta}/x$  (cf. [12], p. 143-144). Assume now that  $\delta > 0$ . Then  $X$  can be embedded in  $Z = B_S$ , where  $S_t = \delta^2 t + V_t$ . The exactly sampling of the first passage event of  $S$  has been shown in [7]. Then Algorithm 1 can be used to sample  $(\varsigma, X_{\varsigma-}, X_{\varsigma})$ .

## 4 Sampling of first exit time of a Brownian motion

In this section, denote

$$\psi(x) = xe^{-x^2/2}, \quad C_0 = 2/\sqrt{e}.$$

Then  $xe^{-xy/2} \leq C_0 e^{-y/2}$  for all  $x, y \geq 1$ , in particular,

$$\psi(xy) = xy e^{-x^2 y^2 / 2} \leq C_0 y e^{-xy^2 / 2}. \quad (4.1)$$

For  $s > 0$  and  $k \geq 0$ , denote

$$d_k(s) = \psi((4k+1)\sqrt{2s}) - \psi((4k+3)\sqrt{2s}). \quad (4.2)$$

Given  $a > 0$ , for  $t > 0$ , by (2.3),

$$p_a(t, 0) = \frac{\pi}{2a^2\sqrt{2x}} \sum_{k=0}^{\infty} d_k(x), \quad \text{with } x = \frac{\pi^2 t}{8a^2}, \quad (4.3)$$

and by (2.2),

$$p_a(t, 0) = \sqrt{\frac{2}{\pi}} \frac{2y}{a^2} \sum_{k=0}^{\infty} d_k(y), \quad \text{with } y = \frac{a^2}{2t}. \quad (4.4)$$

Let  $X = \pi^2 \eta_a / (8a^2)$ . Then by (4.3), the p.d.f. of  $X$  is

$$\begin{aligned} f_X(x) &= (8a^2/\pi^2) \times p_a(8a^2 x / \pi^2) \\ &= A \times \frac{e^{-x}}{\mathbb{P}\{\xi \geq \pi^2/8\}} \times \frac{1 - e^{-\pi^2/2}}{1 - e^{-4x}} \sum_{k=0}^{\infty} (1 - e^{-4x}) e^{-4kx} \times \frac{d_k(x)}{C_0 \sqrt{2x} e^{-(4k+1)x}}, \end{aligned} \quad (4.5)$$



where  $\xi \sim \text{Gamma}(1)$  and

$$A = \frac{4C_0 \mathbb{P}\{\xi \geq \pi^2/8\}}{\pi(1 - e^{-\pi^2/2})}. \quad (4.6)$$

For  $t \geq a^2$ ,  $x \geq \pi^2/8$ , so  $e^{-x}/e^{-\pi^2/8}$  is the p.d.f. of  $\xi$  at  $x$  conditional on  $\xi \geq \pi^2/8$  and  $(1 - e^{-\pi^2/2})/(1 - e^{-4x}) \leq 1$ . Next,  $(1 - e^{-4x})e^{-4kx} = \mathbb{P}\{\kappa = k\}$  for  $\kappa \sim \text{Geo}(e^{-4x})$ . Finally, since  $\sqrt{2x} > 1$  and  $\psi$  is positive and strictly decreasing on  $[1, \infty)$ ,

$$0 < \frac{d_k(x)}{C_0 \sqrt{2x} e^{-(4k+1)x}} < 1,$$

where the second inequality uses (4.1). As a result, (4.5) implies that rejection sampling can be used to sample  $X$  conditional on  $X \geq \pi^2/8$ , and hence to sample  $\eta_a$  conditional on  $\eta_a \geq a^2$ .

Likewise, by (4.4), the p.d.f. of  $Y = a^2/(2\eta_a)$  is

$$\begin{aligned} f_Y(y) &= a^2/(2y^2) \times p_a(a^2/(2y), 0) \\ &= B \times \frac{e^{-y}/\sqrt{\pi y}}{\mathbb{P}\{\zeta > 1/2\}} \times \frac{1 - e^{-2}}{1 - e^{-4y}} \sum_{k=0}^{\infty} (1 - e^{-4y})e^{-4ky} \times \frac{d_k(y)}{C_0 \sqrt{2y} e^{-(4k+1)y}}, \end{aligned} \quad (4.7)$$

where  $\zeta \sim \text{Gamma}(1/2)$  and

$$B = \frac{2C_0 \mathbb{P}\{\zeta > 1/2\}}{1 - e^{-2}}. \quad (4.8)$$

As a result, rejection sampling can be used to sample  $Y$  conditional on  $Y \geq 1/2$ , and hence to sample  $\eta_a$  conditional on  $\eta_a \leq a^2$ .

Recall that  $P^x$  denotes probability measure under which  $B_0 \equiv x$ . The above results lead to the rejection sampling Algorithm 2.

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**Algorithm 2** Sampling  $\eta_a$  under  $P^0$

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**Require:**  $a \in (0, \infty)$ ,  $C_0 = 2/\sqrt{e}$ ,  $d_k(\cdot)$  as in (4.2),  $A$  and  $B$  as in (4.6) and (4.8), respectively

```

1: while (1) do
2:   Sample  $U_1, \dots, U_5$  i.i.d.  $\sim \text{Unif}(0, 1)$ 
3:   if  $U_1 \leq A/(A + B)$  then
4:     Sample  $\xi \sim \text{Gamma}(1)$  conditional on  $\xi \geq \pi^2/8$ , then sample  $\kappa \sim \text{Geo}(e^{-4\xi})$ 
5:     if  $U_2(1 - e^{-4\xi}) \leq 1 - e^{-\pi^2/2}$  and  $C_0 U_3 \sqrt{2\xi} e^{-(4\kappa+1)\xi} \leq d_\kappa(\xi)$  then
6:       return  $8a^2\xi/\pi^2$ 
7:     end if
8:   else
9:     Sample  $\zeta \sim \text{Gamma}(1/2)$  conditional on  $\zeta > 1/2$ , then sample  $\kappa \sim \text{Geo}(e^{-4\zeta})$ 
10:    if  $U_4(1 - e^{-4\zeta}) \leq 1 - e^{-2}$  and  $C_0 U_5 \sqrt{2\zeta} e^{-(4\kappa+1)\zeta} \leq d_\kappa(\zeta)$  then
11:      return  $a^2/(2\zeta)$ 
12:    end if
13:  end if
14: end while

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## 5 Distribution of pre-exit location of a Brownian motion

We need a few more properties of the first exit event of a Brownian motion. For  $t > 0$  and  $x \in (-a, a)$ , it is known that

$$\mathbb{P}^x\{\eta_a \in dt, B_{\eta_a} = \pm a\} = p_a^\pm(t, x) dt,$$

where

$$p_a^+(t, x) = p_a^-(t, -x) = \sum_{k=0}^{\infty} [f_{4ka+a-x}(t) - f_{4ka+3a+x}(t)] \quad (5.1)$$

([5], p. 212, 3.0.6). We also have the following.

**Proposition 2.** *For all  $t > 0$  and  $x \in (-a, a)$*

$$p_a^+(t, x) = p_a^-(t, -x) = \frac{1}{2}p_a(t, x) - \frac{\pi}{2a^2} \sum_{k=1}^{\infty} (-1)^k k \exp\left\{-\frac{k^2\pi^2 t}{2a^2}\right\} \sin \frac{k\pi x}{a}. \quad (5.2)$$

Furthermore,  $p_a^\pm(t, x) > 0$ .

*Proof.* To start with, by the Markov property of  $B$ ,

$$\mathbb{P}^x\{\eta_a > t, B_{\eta_a} = a\} = \mathbb{E}^x[\mathbf{1}\{t < \eta_a\} \mathbf{1}\{B_{\eta_a} = a\}] = \mathbb{E}^x[\mathbf{1}\{t < \eta_a\} \mathbb{P}^{B_t}\{B'_{\eta_a} = a\}],$$

where  $B'$  is an i.i.d. copy of  $B$ . By  $\mathbb{P}^y(B_{\eta_a} = a) = (a + y)/(2a)$ ,  $y \in (-a, a)$  ([5], p. 212, 3.0.4),

$$\mathbb{P}^x\{\eta_a > t, B_{\eta_a} = a\} = \mathbb{E}^x[\mathbf{1}\{t < \eta_a\} (a + B_t)/(2a)] = \frac{1}{2}\mathbb{P}^x\{t < \eta_a\} + \frac{1}{2a}\mathbb{E}^x[B_t \mathbf{1}\{t < \eta_a\}].$$

Put  $u(t, x) = \mathbb{E}^x[B_t \mathbf{1}\{t < \eta_a\}]$ . Using (2.4) and the fact that  $x$  is antisymmetric on  $(-a, a)$ ,

$$u(t, x) = \sum_{k=1}^{\infty} \alpha_k \exp\left\{-\frac{k^2\pi^2 t}{2a^2}\right\} \sin \frac{k\pi x}{a},$$

where

$$\alpha_k = \frac{1}{a} \int_{-a}^a x \sin \frac{k\pi x}{a} dx = (-1)^{k-1} \frac{2a}{k\pi}.$$

Then

$$\mathbb{P}^x\{\eta_a > t, B_{\eta_a} = a\} = \frac{1}{2}\mathbb{P}^x\{\eta_a > t\} - \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \exp\left\{-\frac{k^2\pi^2 t}{2a^2}\right\} \sin \frac{k\pi x}{a}.$$

Differentiating both sides in  $t$  then yields (5.2).

To show  $p_a^+(t, x) > 0$  for all  $x \in (-a, a)$ , regard  $x$  as a parameter while  $t$  the only variate. Then denote  $p_{x,a}^\pm(t) = p_a^\pm(t, x)$ . By (5.1),  $p_{x,a}^+ = f_{a-x} - f_{2a} * p_{x,a}^-$ ,  $p_{x,a}^- = f_{a+x} - f_{2a} * p_{x,a}^+$ . Then  $p_{x,a}^- \leq f_{a+x}$ , so  $p_{x,a}^+ \geq f_{a-x} - f_{2a} * f_{a+x} = f_{a-x} - f_{3a+x}$ . From (2.1), it follows that  $p_{x,a}^+(t) > 0$  for all small  $t > 0$ . Likewise,  $p_{x,a}^-(t) > 0$  for all small  $t > 0$ . Assume  $\{t > 0 : p_{x,a}^+(t) = 0\} \neq \emptyset$  and let  $t_0$  be the infimum of the set. Then  $t_0 > 0$ . Since  $p_{x,a}^+$  is smooth,  $p_{x,a}^+(t_0) = 0$ . Fix  $c \in (x, a)$ . Under  $\mathbb{P}^x$ , in order for  $B$  to reach  $a$  before reaching  $-a$ , it may first reach  $c$  before reaching  $-a$ , then starting at  $c$ , return to  $x$  before reaching  $a$ , and finally, starting at  $x$  again, reach  $a$  before reaching  $-a$ . As a result,  $p_{x,a}^+ \geq p_{x',a'}^+ * p_{x'',a''}^- * p_{x,a}^+$ , where  $x' = x + (a - c)/2$ ,  $a' = (c + a)/2$ ,  $x'' = c - (a + x)/2$ ,  $a'' = (a - x)/2$ . Since  $p_{x,a}^+(t) > 0$  for all  $0 < t < t_0$ , it follows that  $p_{x',a'}^+ * p_{x'',a''}^-(t) = 0$  for all  $t < t_0$ . However  $p_{x',a'}^+(t) > 0$  and  $p_{x'',a''}^-(t) > 0$  for all small  $t$ . The contradiction shows  $p_{x,a}^+$  is strictly positive. Likewise,  $p_{x,a}^-$  is strictly positive.  $\square$

**Proposition 3.** Fix  $t > 0$  and  $a > 0$ . Then for  $x \in (-a, a)$ ,

$$p_a^+(t, x) \leq c_p \min\{\phi_t(a-x)(a-x), \phi_t(a-|x|)(a+x)\}, \quad (5.3)$$

where

$$c_p = 2t^{-1} \sum_{k=0}^{\infty} \left[ \frac{(2ka+4a)^2}{t} + 1 \right] e^{-2k^2a^2/t}. \quad (5.4)$$

Furthermore,

$$\partial_x p_a^+(t, a) := \lim_{x \rightarrow a-} \partial_x p_a^+(t, x) = -\frac{\pi^2}{8a^3} \sum_{k=1}^{\infty} k^2 \exp \left\{ -\frac{k^2 \pi^2 t}{8a^2} \right\} \in (-\infty, 0) \quad (5.5)$$

and

$$\partial_x p_a^+(t, -a) := \lim_{x \rightarrow (-a)+} \partial_x p_a^+(t, x) = \frac{\pi^2}{8a^3} \sum_{k=1}^{\infty} (-1)^{k-1} k^2 \exp \left\{ -\frac{k^2 \pi^2 t}{8a^2} \right\} \in (0, \infty). \quad (5.6)$$

*Proof.* Recall  $tf_y(t) = y\phi_t(y)$ . Let  $y = a - x$ . Then by (5.1),  $p_a^+(t, x) = f_y(t) - t^{-1}g(y)$ , where

$$g(y) = t \sum_{k \geq 2 \text{ even}} [f_{2ka-y}(t) - f_{2ka+y}(t)].$$

By  $g(0) = 0$ ,  $g(y) = g'(\theta y)y$  for some  $\theta = \theta(y) \in (0, 1)$ . Then  $p_a^+(t, x) \leq t^{-1}y[\phi_t(y) + |g'(\theta y)|]$ . By  $t\partial_y f_y(t) = -(y^2/t - 1)\phi_t(y)$ ,

$$|g'(\theta y)| \leq \sum_{k \geq 2 \text{ even}} \left\{ \left[ \frac{(2ka - \theta y)^2}{t} + 1 \right] \phi_t(2ka - \theta y) + \left[ \frac{(2ka + \theta y)^2}{t} + 1 \right] \phi_t(2ka + \theta y) \right\}.$$

Since  $y \in (0, 2a)$ , for  $k \geq 2$ ,  $2ka - \theta y \geq 2(k-2)a + y > 0$  and  $2ka + \theta y \geq 2(k-1)a + y > 0$ . Also, for any  $u \geq 0$ ,  $\phi_t(u+y) \leq e^{-u^2/(2t)}\phi_t(y)$ . Then

$$\begin{aligned} |g'(\theta y)| &\leq \sum_{k \geq 2 \text{ even}} \left\{ \left[ \frac{(2ka)^2}{t} + 1 \right] \phi_t(2(k-2)a + y) + \left[ \frac{(2ka+2a)^2}{t} + 1 \right] \phi_t(2(k-1)a + y) \right\} \\ &\leq \sum_{k \geq 2 \text{ even}} \left\{ \left[ \frac{(2ka)^2}{t} + 1 \right] e^{-2(k-2)^2a^2/t} + \left[ \frac{(2ka+2a)^2}{t} + 1 \right] e^{-2(k-1)^2a^2/t} \right\} \phi_t(y). \end{aligned}$$

It follows that  $p_a^+(t, x) \leq c_p y \phi_t(y) = c_p (a-x) \phi_t(a-x)$ .

Let  $z = a + x$ . Then by (5.1),  $p_a^+(t, x) = t^{-1}h(z)$ , where

$$h(z) = t \sum_{k \geq 1 \text{ odd}} [f_{2ka-z}(t) - f_{2ka+z}(t)].$$

Since  $h(0) = 0$ , there is  $\theta = \theta(z) \in (0, 1)$  such that  $h(z) = h'(\theta z)z$ . Similar to the above argument,

$$\begin{aligned} |h'(\theta z)| &\leq \sum_{k \geq 1 \text{ odd}} \left\{ \left[ \frac{(2ka - \theta z)^2}{t} + 1 \right] \phi_t(2ka - \theta z) + \left[ \frac{(2ka + \theta z)^2}{t} + 1 \right] \phi_t(2ka + \theta z) \right\} \\ &= \left[ \frac{(2a - \theta z)^2}{t} + 1 \right] \phi_t(2a - \theta z) + I_1 + I_2 \\ &\leq [(2a)^2/t + 1] \phi_t(2a - z) + I_1 + I_2, \end{aligned}$$

where

$$I_1 = \sum_{k \geq 3 \text{ odd}} \left[ \frac{(2ka - \theta z)^2}{t} + 1 \right] \phi_t(2ka - \theta z) \leq \sum_{k \geq 3 \text{ odd}} \left[ \frac{(2ka)^2}{t} + 1 \right] \phi_t(2(k-2)a + z),$$

$$I_2 = \sum_{k \geq 1 \text{ odd}} \left[ \frac{(2ka + \theta z)^2}{t} + 1 \right] \phi_t(2ka + \theta z) \leq \sum_{k \geq 1 \text{ odd}} \left[ \frac{(2ka + 2a)^2}{t} + 1 \right] \phi_t(2(k-1)a + z).$$

Then  $p_a^+(a, x) \leq t^{-1}[(2a)^2/t + 1]\phi_t(a - x)z + (c_p/2)\phi_t(z)z \leq c_p\phi_t(a - |x|)(a + x)$ .

The equalities in (5.5) and (5.6) can be shown by direct calculation using (2.3) and (5.2). Then it is clear that  $\partial_x p_a^+(t, a)$  is strictly negative and finite and  $\partial_x p_a^+(t, -a)$  is finite. To show that the latter partial derivative is strictly positive, first, write it as

$$\frac{\pi^2}{8a^3} \sum_{k \geq 1 \text{ odd}} [g_s(k^2) - g_s((k+1)^2)],$$

where  $g_s(x) = x \exp\{-sx\}$  with  $s = \pi^2 t / 8a^2$ . If  $s \geq 1$ , i.e.,  $t \geq 8a^2/\pi^2$ , then  $g_s(x)$  is strictly decreasing on  $[1, \infty)$  and hence  $\partial_x p_a^+(t, -a) > 0$ . To cover the case  $0 < t < 8a^2/\pi^2$ , differentiate (5.1) instead. Using the expressions in (2.1),

$$\partial_x p_a^+(t, -a) = \frac{2}{\sqrt{2\pi}t^{3/2}} \sum_{k \geq 1 \text{ odd}} h_s(k^2),$$

where  $h_s(x) = (sx - 1)e^{-sx/2}$  with  $s = 4a^2/t$ . It is seen that if  $0 < t < 4a^2$ , in particular, if  $0 < t < 8a^2/\pi^2$ , then the sum on the right hand side is strictly positive.  $\square$

From the proof of (5.3) it is seen that the following is actually true.

**Corollary 4.** Fix  $t > 0$  and  $a > 0$ . Let  $c_p$  be as in (5.4). Then for  $x \in (-a, a)$  and  $n \geq 0$ ,

$$\left| \sum_{k=0}^n [f_{4ka+a-x}(t) - f_{4ka+3a+x}(t)] + f_{4(n+1)a+a-x}(t) \right| \leq c_p \phi_t(a - x)(a - x)$$

and

$$\left| \sum_{k=0}^n [f_{4ka+a-x}(t) - f_{4ka+3a+x}(t)] \right| \leq c_p \phi_t(a - |x|)(a + x).$$

**Proposition 5.** Given  $a > 0$ ,

$$\mathbf{P}^0\{B_t \in dx, \eta_a > t\} = \mathbf{1}\{|x| < a\} q_a(t, x) dx,$$

where

$$q_a(t, x) = \sum_{k=-\infty}^{\infty} (-1)^k \phi_t(x - 2ka) \tag{5.7}$$

$$= \frac{1}{a} \sum_{k=0}^{\infty} \exp\left\{-\frac{(2k+1)^2 \pi^2}{8a^2} t\right\} \cos \frac{(2k+1)\pi x}{2a}. \tag{5.8}$$

In addition, for all  $x \in (-a, a)$ ,  $q_a(t, x) > 0$  and

$$q_a(t, x) \leq c_q \phi_t(x)(a - |x|) \quad (5.9)$$

where  $c_q = 8at^{-1} \sum_{k=0}^{\infty} (k+1)e^{-2k^2a^2/t}$ . Finally,

$$q_a(t, \pm a) = 0, \quad \partial_x q_a(t, a) = -\partial_x q_a(t, -a) = -p_a(t, 0) < 0. \quad (5.10)$$

*Proof.* The assertion is trivial if  $|x| \geq a$ . For  $|x| < a$ ,

$$P^0\{B_t \in dx, \eta_a > t\} = P^0\{B_t \in dx\} - P^0\{B_t \in dx, \eta_a \leq t\}.$$

By the strong Markov property,

$$\begin{aligned} P^0\{B_t \in dx, \eta_a \leq t, B_{\eta_a} = a\} &= \int_0^t P^0\{B_t \in dx \mid \eta_a = s, B_s = a\} P^0\{\eta_a \in ds, B_{\eta_a} = a\} \\ &= \frac{1}{2} \int_0^t P^0\{B_{t-s} \in dx - a\} p_a(s, 0) ds. \end{aligned}$$

By (2.2) and the strong Markov property again,

$$\begin{aligned} P^0\{B_t \in dx, \eta_a \leq t, B_{\eta_a} = a\} &= \sum_{k=1}^{\infty} (-1)^{k-1} \int_0^t P^0\{B_{t-s} \in dx - a\} f_{(2k-1)a}(s) ds \\ &= \sum_{k=1}^{\infty} (-1)^{k-1} P^0\{B_t \in dx + 2(k-1)a, \tau_{(2k-1)a} \leq t\} = \sum_{k=1}^{\infty} (-1)^{k-1} P^0\{B_t \in 2ka - dx\}, \end{aligned}$$

where the last equality is due to the reflection principle. For each  $k$ ,  $P^0\{B_t \in 2ka - dx\} = \phi_t(x - 2ka) dx$ . A similar formula holds for  $P^0\{B_t \in dx, \eta_a \leq t, B_{\eta_a} = -a\}$ . Combining the results, (5.7) follows. By (2.4), for any  $g \in C([-a, a])$ ,

$$\begin{aligned} \int_{-a}^a g(x) P^0(B_t \in dx, t < \eta_a) &= E^0[g(B_t) \mathbf{1}\{t < \eta_a\}] \\ &= \sum_{k \geq 1, k \text{ odd}} \frac{1}{a} \exp\left\{-\frac{k^2 \pi^2 t}{8a^2}\right\} \int_{-a}^a g(x) \cos \frac{k\pi x}{2a} dx. \end{aligned}$$

Comparing the measures on both sides yields (5.8).

To show  $q_a(t, x) > 0$  for  $|x| < a$ , fix  $s \in (0, t)$  and  $0 < \delta < (a - |x|)/2$ . Then  $(x - \delta, x + \delta) \subset (-a, a)$ , so by the Markov property,

$$\begin{aligned} P^0\{B_t \in dx, \eta_a > t\} &\geq \int_{x-\delta}^{x+\delta} P^0\{B_s \in du, \eta_a > s\} P^u\{B_{t-s} \in dx, \eta_a > t-s\} \\ &\geq \int_{x-\delta}^{x+\delta} P^0\{B_s \in du, \eta_a > s\} P^0\{B_{t-s} \in dx - u, \eta_\delta > t-s\}, \end{aligned}$$

where the second inequality is due to  $[-\delta, \delta] \subset [-a-u, a-u]$  for all  $u \in [x-\delta, x+\delta]$ . As a result,

$$q_a(t, x) \geq \int_{x-\delta}^{x+\delta} q_a(s, u) q_\delta(t-s, x-u) du.$$

If  $q_a(t, x) = 0$ , then  $\tilde{q}(u) := q_a(s, u)q_\delta(t - s, x - u) = 0$  for all  $u \in (x - \delta, x + \delta)$ . However, by (5.7),  $\tilde{q}$  is analytic on  $\mathbb{C}$ . This leads to  $\tilde{q}(u) \equiv 0$ , implying either  $q_a(s, \cdot) \equiv 0$  or  $q_\delta(t - s, \cdot) \equiv 0$ , which is impossible, as  $q_a(t, 0) > 0$  by (5.8). The contradiction shows  $q_a(t, x) > 0$ .

To show (5.9), since  $q_a(t, x) = q_a(t, -x)$ , it suffices to consider  $x \geq 0$ . Let  $z = a - x$ . Then  $z \in (0, a)$  and by (5.8),

$$q_a(t, x) = \sum_{k=0}^{\infty} (-1)^k [\phi_t(2ka + a - z) - \phi_t(2ka + a + z)] := g(z).$$

Since  $g(0) = 0$ , there is  $\theta = \theta(z) \in (0, 1)$ , such that  $g(z) = g'(\theta z)z$ . By  $\phi'_t(x) = -f_x(t)$

$$\begin{aligned} |g'(\theta z)| &\leq 2 \sum_{k=0}^{\infty} [f_{2ka+a-z}(t) + f_{2ka+a+z}(t)] \\ &\leq 2t^{-1} \sum_{k=0}^{\infty} (2ka + a + z) [\phi_t(2ka + a - z) + \phi_t(2ka + a + z)] \\ &\leq 8at^{-1} \sum_{k=0}^{\infty} (k+1) \phi_t(2ka + a - z) \\ &\leq 8at^{-1} \sum_{k=0}^{\infty} (k+1) e^{-2k^2 a^2/t} \phi_t(a - z). \end{aligned}$$

Then (5.9) follows.

It is straightforward to check the first equation in (5.10). By differentiating each term in (5.8) at  $\pm a$  and comparing the resulting series to (2.3), the second equation in (5.10) obtains. The inequality in (5.10) is just the last assertion of Proposition 2.  $\square$

**Corollary 6.** *Given  $a > 0$  and  $T > 0$ , for  $t > 0$  and  $x \in (-a, a)$ ,*

$$\mathbb{P}^0\{B_T \in dx, \eta_a \in T + dt, B_{\eta_a} = \pm a\} = q_a(T, x)p_a^+(t, \pm x) dx dt. \quad (5.11)$$

*As a result, under  $\mathbb{P}^0$ , conditional on  $\eta_a = T + t$  and  $B_{\eta_a} = a$ , the p.d.f. of  $B_T$  at  $x \in (-a, a)$  is in proportion to  $q_a(T, x)p_a^+(t, x)$  and conditional on  $\eta_a = T + t$  and  $B_{\eta_a} = -a$ , the p.d.f. of  $B_T$  at  $x \in (-a, a)$  is in proportion to  $q_a(T, -x)p_a^+(t, -x)$ .*

*Proof.* Since  $\{\eta_a > T\} \in \mathcal{F}(B_s, s \leq T)$ , the Markov property of  $B$  yields  $\mathbb{P}^0\{B_T \in dx, \eta_a \in T + dt, B_{\eta_a} = a\} = \mathbb{P}^0\{B_T \in dx, \eta_a > T\} \mathbb{P}^x\{\eta_a \in dt, B_{\eta_a} = a\}$ . Then the case  $B_{\eta_a} = a$  of (5.11) follows from Propositions 2 and 5. The case  $B_{\eta_a} = -a$  is similarly proved. Finally, note  $q_a(T, x) = q_a(T, -x)$ .  $\square$

## 6 Sampling of pre-exit location of a Brownian motion

Fix  $t > 0$ ,  $T > 0$ , and  $a > 0$ . The objective now is to sample  $B_t$  conditional on  $\eta_a = T + t$  and  $B_{\eta_a} = \pm a$ . By symmetry, it suffice to consider the case where  $B_{\eta_a} = a$ . We shall construct an envelope function for  $p_a^+(t, x)$  and one for  $q_a(T, x)$ , respectively. Consider  $p_a^+(t, x)$  first. To emphasize that  $x$  is the (only) variate, write  $g_t(x) = f_x(t)$ . Let

$$P_k(x) = g_t(4ka + a - x) - g_t(4ka + 3a + x).$$

By (5.1), for each  $x \in (-a, a)$ ,

$$p_a^+(t, x) = \sum_{k=0}^{\infty} P_k(x). \quad (6.1)$$

Let

$$p^* = p^*(x, t) = \min \left\{ n \geq 0 : \sum_{k=0}^n P_k(x) \geq 0 \right\}. \quad (6.2)$$

Since  $p_a^+(t, x) > 0$  (Proposition 2), the set on the right hand side is nonempty, so  $p^*$  is well defined.

**Proposition 7.** Fix  $t > 0$  and  $a > 0$ . Then for  $x \in (-a, a)$ , 1)  $P_{p^*}(x) \geq 0$  and  $P_k(x) > 0$  for all  $k > p^*$ , 2) as  $x \rightarrow a$ ,

$$p^* \sim \sqrt{\frac{t}{8a^2} \ln \frac{1}{a-x}},$$

and 3) as  $x \rightarrow -a$ ,  $p^* = O(1)$ .

*Proof.* 1) By definition of  $p^*$ , for  $x \in (-a, a)$ ,  $P_{p^*}(x) \geq 0$ , i.e.,  $g_t(4p^*a + a - x) \geq g_t(4p^*a + 3a + x)$ . Since  $g_t(x)$  is strictly increasing on  $(0, \sqrt{t}]$  and strictly decreasing on  $[\sqrt{t}, \infty)$ ,  $4p^*a + 3a + x > \sqrt{t}$ . Then for  $k > p^*$ ,  $4ka + a - x > 4p^*a + 3a + x > \sqrt{t}$ , so  $P_k(x) > 0$ .

2) Put  $y = a - x$ . By Corollary 4

$$g_t(4p^*a + 4a - y) \leq \sum_{0 \leq k < p^*} P_k(a - y) + g_t(4p^*a + y) \leq c_p \phi_t(y)y. \quad (6.3)$$

On the other hand, by definition of  $p^*$ ,

$$u(y) := \sum_{0 \leq k < p^*-1} P_k(a - y) + g_t(4(p^* - 1)a + y) < g_t(4p^*a - y).$$

Since  $u(0) = 0$ , there is  $\theta = \theta(y) \in (0, 1)$  such that  $u(y) = u'(\theta y)y$ . As  $y \rightarrow 0$ ,  $p^* \rightarrow \infty$  by (6.3). Then by Proposition 3 and uniform convergence,  $u'(\theta y) \rightarrow d := -\partial_x p_a^+(t, a) > 0$ . As a result,  $(1 + o(1))dy \leq g_t(4p^*a - y)$ . This combined with (6.3) yields the claimed asymptotic of  $p^*$ .

3) Put  $z = a + x$  and

$$v_n(z) = \frac{1}{t} \sum_{k=0}^n [g_t(4ka + 2a - z) - g_t(4ka + 2a + z)]$$

if  $n \geq 0$  and  $v_n(z) = 0$  if  $n < 0$ . Since  $v_n(0) = 0$ , there is  $\theta_n = \theta_n(z) \in (0, 1)$  such that  $v_n(z) = v'_n(\theta_n z)z$ . It is not hard to show that as  $z \rightarrow 0$ ,  $\sup_n |v'_n(\theta_n z) - d_n| \rightarrow 0$ , where  $d_n = -2t^{-1} \sum_{0 \leq k < n} g'_t(4ka + 2a)$ . Since  $d_\infty = \partial_x p_a^+(t, -a)$ , which by Proposition 3 is strictly positive, there is  $n^*$  such that  $d_n > 0$  for all  $n \geq n^*$ . Since  $v_{p^*-1}(z) \leq 0$ , this implies  $p^* - 1 < n^*$  for all  $z > 0$  small enough. Thus  $p^* = O(1)$ .  $\square$

The asymptotics of  $p^*$  in Proposition 7 suggest that for rejection sampling involving  $p_a^+(t, x)$ ,  $x \approx a$  should be handled more carefully than  $x \approx -a$ . Letting  $y = a - x$ , for  $k \geq 1$ ,

$$\begin{aligned} P_{p^*+k}(x) &\leq g_t(4(p^* + k)a + y) \\ &\leq g_t(4(p^* + 1)a + y) \times \frac{4(p^* + k)a + y}{4(p^* + 1)a + y} e^{-8(k-1)^2 a^2 / t}, \end{aligned}$$



where the second inequality is due to  $\phi_t(y+z) \leq \phi_t(y)e^{-z^2/(2t)}$  for all  $y, z > 0$ . From the proof of Proposition 7,  $4(p^*+1)a - y \geq \sqrt{t}$  and  $\phi(x)$  is strictly decreasing on  $[\sqrt{t}, \infty)$ . Thus  $g_t(4(p^*+1)a+y) < g_t(4(p^*+1)a-y)$  and by (6.3),

$$0 < P_{p^*+k}(x) \leq c_p \phi_t(a-x)(a-x) \times k e^{-8(k-1)^2 a^2/t} \quad (6.4)$$

and meanwhile

$$0 \leq \sum_{k=0}^{p^*} P_k(x) \leq c_p \phi_t(a-x)(a-x). \quad (6.5)$$

We will use (6.1)–(6.5) to construct an envelope for  $p_a^+(t, x)$ .

To construct an envelope for  $q_a(T, x)$ , for each  $x \in (-a, a)$ , the series in (5.7) converges absolutely. Let

$$Q_k(x) = \phi_T(4ka + |x|) - \phi_T(4ka + 2a - |x|) - \phi_T(4ka + 2a + |x|) + \phi_T(4ka + 4a - |x|).$$

Noting  $q_a(T, x) = q_a(T, -x)$ ,

$$q_a(T, x) = \sum_{k=0}^{\infty} Q_k(x). \quad (6.6)$$

Let

$$q^* = q^*(x, T) = \min \left\{ n \geq 0 : \sum_{k=0}^n Q_k(x) \geq 0 \right\}. \quad (6.7)$$

Since  $q_a(T, x) > 0$  by Proposition 5,  $q^*$  is well defined.

**Proposition 8.** Fix  $T > 0$  and  $a > 0$ . Then for  $x \in (-a, a)$ , 1)  $Q_{q^*}(x) \geq 0$ ,  $Q_k(x) > 0$  for all  $k > q^*$ , and 2) as  $|x| \rightarrow a$ ,  $q^* = O(1)$ .

Indeed, since  $\phi_T$  is strictly concave on  $(0, \sqrt{T}]$  and strictly convex on  $[\sqrt{T}, \infty)$ , 1) follows by similar argument for 1) of Proposition 7. On the other hand, 2) follows from (5.10) and similar argument for 3) of Proposition 7. The detail of the proof is omitted for brevity.

Since  $\phi_T$  is strictly decreasing on  $[0, \infty)$ , for each  $k \geq 0$ ,  $Q_k(x) < \phi_T(4ka + |x|) - \phi_T(4ka + 2a - |x|) = -2\phi'_T(4ka + y)(a - |x|)$ , where  $y = y(x, k) \in (|x|, 2a - |x|)$ . By  $|\phi'_T(4ka + y)| \leq T^{-1}(4ka + 2a)\phi_T(4ka + |x|) \leq T^{-1}(4ka + 2a)e^{-8k^2 a^2/T}\phi_T(x)$ ,

$$Q_k(x) \leq 4aT^{-1}(2k+1)e^{-8k^2 a^2/T}\phi_T(x)(a - |x|), \quad k > q^*. \quad (6.8)$$

Meanwhile,

$$0 \leq \sum_{k=0}^{q^*} Q_k(x) < 4aT^{-1}(2q^*+1)e^{-8(q^*)^2 a^2/T}\phi_T(x)(a - |x|). \quad (6.9)$$

Now (6.1)–(6.9) can be combined as follows. Define

$$\gamma(x) = \gamma(x, T, t) = 4aT^{-1}\mathbf{1}\{|x| < a\} \phi_t(a-x)(a-x)\phi_T(x)(a - |x|). \quad (6.10)$$

For  $k \geq 0$ , define

$$a_k = a_k(t) = \begin{cases} 1 & k = 0 \\ 2ke^{-8(k-1)^2 a^2/t} & k \geq 1 \end{cases} \quad (6.11)$$

and

$$b_k = b_k(T) = (2k+1)e^{-8k^2 a^2/T}. \quad (6.12)$$

For  $x \in (-a, a)$  and  $k \geq 0$ , define

$$r_k(x, m) = \begin{cases} \frac{\sum_{k=0}^m P_k(x)}{c_p \phi_t(a-x)(a-x)a_k} & \text{if } k = 0 \\ \frac{P_{m+k}(x)}{c_p \phi_t(a-x)(a-x)a_k} & \text{else} \end{cases} \quad (6.13)$$

and

$$s_k(x, m) = \begin{cases} 0 & \text{if } k < m \\ \frac{\sum_{k=0}^m Q_k(x)}{4aT^{-1}\phi_T(x)(a-|x|)b_k} & \text{if } k = m \\ \frac{Q_k(x)}{4aT^{-1}\phi_T(x)(a-|x|)b_k} & \text{else} \end{cases} \quad (6.14)$$

Then  $r_k(x, p^*) \in [0, 1)$ ,  $s_k(x, q^*) \in [0, 1)$  for all  $k \geq 0$  and

$$p_a^+(t, x)q_a(T, x) = \gamma(x) \times \sum_{k=0}^{\infty} a_k r_k(x, p^*) \times \sum_{k=0}^{\infty} b_k s_k(x, q^*),$$

The rejection sampling of the p.d.f.  $\gamma(x)/\int_{-a}^a \gamma$  is quite routine, although for efficiency, the detail has to depend on  $a$ ,  $t$ , and  $T$  (more precisely, on  $a/\sqrt{t}$  and  $a/\sqrt{T}$ ). On the other hand, note that both  $a_k$  and  $b_k$  are log-concave, i.e.,  $a_{k-1}a_{k+1} \geq a_k^2$  and  $b_{k-1}b_{k+1} \geq b_k^2$ . It was shown in [8] that a log-concave distribution on integers can be sampled efficiently by rejection sampling. The only minor issue here is that the values of the normalizing constants for  $a_k$  and  $b_k$  in general are not available exactly. However, it is easy to find positive lower and upper bounds for the sequences. When these bounds are used in place of the exact normalizing constants, the rejection sampling in [8] still works with some minor modifications and loss of efficiency.

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**Algorithm 3** Sampling  $B_T$  under  $P^0$ , conditional on  $\eta_a = T + t$  and  $B_{\eta_a} = a$

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**Require:**  $a \in (0, \infty)$ ,  $T \in (0, \infty)$ ,  $t \in (0, \infty)$

---

1: **repeat**

2:   Sample  $X \sim \gamma/\int \gamma$ , where  $\gamma(x) = \gamma(x, T, t)$  is defined in (6.10)

3:   Compute  $p^* = p^*(X, t)$  by (6.2) and  $q^* = q^*(X, T)$  by (6.7)

4:   Sample  $\kappa_1 \in \{0, 1, \dots\}$  with p.m.f.  $a_k/\sum_j a_j$ , where  $a_k = a_k(t)$  is defined in (6.11)

5:   Sample  $\kappa_2 \in \{0, 1, \dots\}$  with p.m.f.  $b_k/\sum_j b_j$ , where  $b_k = b_k(T)$  is defined in (6.12)

6:   Sample  $U \sim \text{Unif}(0, 1)$

7: **until**  $r_{\kappa_1}(X, p^*)s_{\kappa_2}(X, q^*) \leq U$ , where  $r_k$  and  $s_k$  are defined in (6.13) and (6.14), respectively

8: **return**  $X$

---

The above results lead to the rejection sampling Algorithm 3.

## 7 Comments

This paper only considers Lévy processes that can be embedded into a subordinated standard Brownian motion. In principle, the scheme illustrated in Figure 1 can be applied to any Lévy process, for example, a spectrally negative  $\alpha$ -stable Lévy process with  $\alpha \in [1, 2)$ . Indeed, without subordination being involved, the scheme can be somewhat simplified. However, even for the spectrally negative  $\alpha$ -stable process, which has many remarkable properties, there are few close form formulas for its first exit event (cf. [2, 9, 15]). Since all Lévy processes are differences of independent spectrally negative Lévy processes, it would be interesting to find exact sampling methods for the first exit event of the latter.

The procedure in Algorithm 1 in principle may also be applied to a subordinated Brownian motion that has a drift or takes values in  $\mathbb{R}^d$  with  $d > 1$ . For example, in each iteration, instead of an interval centered at the current value of a Brownian motion (cf. step 3), use a sphere of small radius so that jumps of large size can be detected and removed. One potential problem is that for  $d > 1$ , when there is a positive chance for the Lévy process to creep across the boundary of a region, the iteration in Algorithm 1 may not be able to stop. This is because the chance for the Brownian motion to hit the “right spots” on the sphere, namely, the intersection between the sphere and the boundary of the region, can be 0; at any other spot on the sphere, a new sphere has to be constructed, possibly with a smaller radius. As a result, the iteration will approach infinitely but never reach the location of the first exit. Regardless, there is no extra work on the sampling for the subordinator. Also, the first exit time out of a sphere by a Brownian motion is well understood. On the other hand, the distribution of the pre-exit value of the Brownian motion becomes substantially subtle. Provided the distribution is available in close form, the procedure of the paper can be extended straightforwardly.

## References

- [1] ALILI, L. and KYPRIANOU, A. E. (2005). Some remarks on first passage of Lévy processes, the American put and pasting principles. *Ann. Appl. Probab.*, **15** 2062–2080.
- [2] BERTOIN, J. (1996). *Lévy processes*, vol. 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge.
- [3] BERTOIN, J. (1997). Regularity of the half-line for Lévy processes. *Bull. Sci. Math.*, **121**.
- [4] BERTOIN, J., VAN HARN, K. and STEUTEL, F. W. (1999). Renewal theory and level passage by subordinators. *Statist. Probab. Lett.*, **45** 65–69.
- [5] BORODIN, A. N. and SALMINEN, P. (2002). *Handbook of Brownian motion—facts and formulae*. 2nd ed. Probability and its Applications, Birkhäuser Verlag, Basel.
- [6] CHI, Z. (2012). On exact sampling of nonnegative infinitely divisible random variables. *Adv. Appl. Probab.*, **44** 842–873.
- [7] CHI, Z. (2016). On exact sampling of the first passage event of a Lévy process with infinite Lévy measure and bounded variation. *Stochastic Processes Appl.*, **126** 1124–1144.
- [8] DEVROYE, L. (1987). A simple generator for discrete log-concave distributions. *Computing*, **39** 87–91.
- [9] DONEY, R. A. (2007). *Fluctuation theory for Lévy processes*, vol. 1897 of *Lecture Notes in Mathematics*. Springer, Berlin.

- [10] DONEY, R. A. and KYPRIANOU, A. E. (2006). Overshoots and undershoots of Lévy processes. *Ann. Appl. Probab.*, **16** 91–106.
- [11] DONEY, R. A. and MALLER, R. A. (2002). Stability of the overshoot for Lévy processes. *Ann. Probab.*, **30** 188–212.
- [12] GLASSERMAN, P. (2004). *Monte Carlo methods in financial engineering*, vol. 53 of *Applications of Mathematics (New York)*. Springer-Verlag, New York. Stochastic Modelling and Applied Probability.
- [13] HUZAK, M., PERMAN, M., ŠIKIĆ, H. and VONDRAČEK, Z. (2004). Ruin probabilities for competing claim processes. *J. Appl. Probab.*, **41** 679–690.
- [14] KLÜPPELBERG, C., KYPRIANOU, A. E. and MALLER, R. A. (2004). Ruin probabilities and overshoots for general Lévy insurance risk processes. *Ann. Appl. Probab.*, **14** 1766–1801.
- [15] KYPRIANOU, A. E. (2006). *Introductory lectures on fluctuations of Lévy processes with applications*. Universitext, Springer-Verlag, Berlin.
- [16] MÖRTERS, P. and PERES, Y. (2010). *Brownian motion*. Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, Cambridge. With an appendix by Oded Schramm and Wendelin Werner.
- [17] SATO, K.-I. (1999). *Lévy processes and infinitely divisible distributions*, vol. 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge. Translated from the 1990 Japanese original, Revised by the author.